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ON RETAINING THE PROPERTY OF UNIFORM ASYMPTOTIC STABILITY WITH RESPECT TO SOME OF THE VARIABLES*

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Two systems of differential equations of perturbed motion (i.e. having a zero solution), with their right-hand sides differing from each other by certain "perturbing" terms, are considered. One of the systems is assumed to be uniformly asymptotically stable with respect to some of the variables. The constraints imposed on perturbing terms are indicated under which the zero solution of the second system retains the property of uniform asymptotic stability with respect to some of the variables.

1. Let the differential equations of perturbed motion have the form

$$x^{\cdot} = X(t, x, y), \quad y^{\cdot} = Y(t, x, y) \tag{1.1}$$

where x, X are the n-dimensional and y, Y the m-dimensional vectors respectively, with components x_i, X_i, y_j, Y_j. We shall consider, together with Eqs.(1.1), the system

$$\begin{aligned} x^{\cdot} &= X(t, x, y) + Q^{(1)}(t, x, y) + R^{(1)}(t, x, y) \\ y^{\cdot} &= Y(t, x, y) + Q^{(2)}(t, x, y) + R^{(2)}(t, x, y) \end{aligned} \tag{1.2}$$

where $Q^{(1)}, R^{(1)}$ are n-dimensional and $Q^{(2)}, R^{(2)}$ m-dimensional vector functions respectively, with components $Q_i^{(1)}, R_i^{(1)}, Q_j^{(2)}, R_j^{(2)}$. In what follows, we shall assume that the right-hand sides of systems (1.1) and (1.2) satisfy the conditions for the existence of unique solutions with prescribed initial conditions, depending continuously on the initial data, in the region

$$\Gamma_H = I \times B_H \times R^m, \quad I = [0, \infty[, \quad B_H = \{x \in R^n : \|x\| < H\} \tag{1.3}$$

We shall assume the solutions of system (1.1) and (1.2) to be y-continuable. This means /1/ that any solution $x(t), y(t)$ will be defined for all $t \geqslant 0$ for which $\|x(t)\| \leqslant H$. Here the symbol $\|\cdot\|$ denotes the Euclidean norm. The functions $R^{(1)}, Q^{(1)}$ satisfy, in the region (1.3), the estimates

$$\|R^{(1)}(t, x, y)\| \leqslant L\|x\|, \ \|Q^{(1)}(t, x, y)\| \leqslant L\|x\| \quad (L = \text{const}),$$

and $R^{(1)}, R^{(2)}$ are assumed to satisfy the Lipshits conditions in x, y. Let

$$\begin{aligned} &z = (x; y), \ Z(t, z) = (X, Y), \ Q(t, z) = (Q^{(1)}; Q^{(2)}) \\ &R(t, z) = (R^{(1)}, R^{(2)}). \end{aligned}$$

Then the conditions will take the form

$$|R_i(t, z^{(1)}) - R_i(t, z^{(2)})| \leqslant L\|z^{(1)} - z^{(2)}\| \quad (i = 1, \ldots, n + m)$$

and Eqs.(1.1) and (1.2) will be written, respectively, as

$$z^{\cdot} = Z(t, z) \tag{1.4}$$
$$z^{\cdot} = Z(t, z) + Q(t, z) + R(t, z). \tag{1.5}$$

Let us assume that systems (1.4) and (1.5) have the trivial solution

$$z = 0 \tag{1.6}$$

The problem of the stability of solution (1.6) with respect to some of the variables has been discussed in a number of papers /1-7/ and a fairly full survey can be found in /7/.

2. We shall introduce the following definitions.

Definition 1. We shall say that the function $Q(t, z)$ satisfies condition A_1 relative to x, if there exists $h > 0$ such that for any $\xi \in]0; h[$ an instant of time $\tau_\xi \geqslant 0$ and a function $g_\xi(t)$ continuous on $[\tau_\xi; \infty[$ can be found such that $|Q_s(t, z)| \leqslant g_\xi(t)$ $(s = 1, \ldots, n + m)$ for all $z \in \overline{(B_h \setminus B_\xi)} \times R^m$, $t \in [\tau_\xi, \infty[$ and

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$$\lim_{t\to\infty} G_\xi(t) = 0, \quad G_\xi(t) = \int_t^{t+1} g_\xi(s)\,ds$$

Definition 2. We shall assume that the function $Q(t, z)$ satisfies the condition A_2 with respect to x, if a number $h > 0$ exists such that if $z \in \bar{B}_h \times R^m$, $t \in I$, then $|Q_s(t, z)| \leqslant g(t)$ $(s = 1, \ldots, n+m)$ and

$$\lim_{t\to\infty} G(t) = 0, \quad G(t) = \int_t^{t+1} g(s)\,ds$$

Clearly, if condition A_2 with respect to x holds, so does condition A_1 with respect to x.

Theorem. Let the solution (1.6) of the system of differential Eqs.(1.4) be uniformly asymptotically x-stable, and let the stability be proved using the twice continuously differentiable function $v(t, z)$, defined in the region Γ_h $(h < H)$ and having, in this region, the estimates

$$a(\|x\|) \leqslant v(t, z) \leqslant b(\|x\|), \quad v^{\cdot}\,|_{(1.4)} \leqslant -c(\|x\|) \tag{2.1}$$
$$\left|\frac{\partial v}{\partial z_i}\right| \leqslant N, \quad \left|\frac{\partial v}{\partial t}\right| \leqslant N, \quad \left|\frac{\partial^2 v}{\partial z_i \partial z_s}\right| \leqslant N, \quad \left|\frac{\partial^2 v}{\partial z_i \partial t}\right| \leqslant N$$

where a, b, c are the Hahn functions and N is a real number. Then, if the function $Q(t, z)$ satisfies condition A_1 with respect to x, and function $R(t, z)$ satisfies the relations

$$\lim_{t\to\infty} \int_t^{t+T} R_i(s, z)\,ds = 0 \tag{2.2}$$

uniformly in $T \in I$, $z \in B_H \times R^m$, then the trivial solution (1.6) of Eq.(1.5) will also be uniformly asymptotically stable with respect to x and $\sigma \in]0; h[$ will exist such, that the set $B_\sigma \times R^m$ will be contained within its domain of x-attraction.

Proof. Let us set an arbitrary $\varepsilon > 0$, assuming that $\varepsilon < h$, and write $\xi = b^{-1}\left[\frac{1}{2} a(\varepsilon)\right]$.
Then by virtue of inequalities (2.1) we have

$$\inf_{\|x\|=\varepsilon} v(t, z) \geqslant a(\varepsilon), \quad \sup_{\|x\|<\xi} v(t, z) \leqslant b(\xi) = \frac{1}{2} a(\varepsilon) \tag{2.3}$$

We shall show that any trajectory $z(t) = z(t, t_1, z_1)$ of Eqs.(1.5) passing, at a sufficiently long instant of time t_1, through an arbitrary point $z_1 = (x_1, y_1)$, $\|x_1\| = \xi$ satisfies, when $t \geqslant t_1$, the inequality $\|x(t)\| < \varepsilon$. Let us assume the opposite. Let a system of equations of the form (1.5) exist, satisfying the conditions formulated above, and let the system have a solution

$$z(t) = z(t, t_1, z_1) \tag{2.4}$$

satisfying the conditions $\|x(t_1)\| = \xi$, $\|x(t_2)\| = \varepsilon$ and, when $t \in [t_1; t_2]$, the trajectory of (2.4) will be situated within the region

$$\xi \leqslant \|x\| \leqslant \varepsilon, \quad y \in R^m$$

By virtue of system(1.5) the derivative of $v(t, z)$ has the form

$$v^{\cdot}\,|_{(1.5)} = v^{\cdot}\,|_{(1.4)} + \sum_{i=1}^{n+m} \frac{\partial v}{\partial z_i}[Q_i(t, z) + R_i(t, z)]$$

and from this we have

$$\Delta v = v(t_2, z(t_2)) - v(t_1, z(t_1)) \leqslant -c(\xi)(t_2 - t_1) + I_Q + I_R; \tag{2.5}$$
$$I_P = \int_{t_1}^{t_2} \sum_{i=1}^{n+m} \frac{\partial v(t, z(t))}{\partial z_i} P_i(t, z(t))\,dt; \quad P = Q, R$$

Let us obtain the estimates for the integrals appearing on the right-hand side of (2.5):

$$|I_Q| \leqslant N(n+m) \int_{t_1}^{t_2} |Q_i(t, z(t))|\,dt \leqslant \tag{2.6}$$
$$N(n+m) \int_{t_1}^{t_2} g_\xi(s)\,ds \leqslant N(n+m) \int_{t_1-1}^{t_2} G_\xi(s)\,ds \quad (t_1 > 1)$$

The last inequality in (2.6) follows from the results of /8/.
Let us introduce the function

$$E_\xi(t) = \sup_{s\in[t-1;\,\infty[} G_\xi(s)$$

which, by virtue of the property $G_\xi(s) \to 0$ as $s \to \infty$, is monotonically non-increasing and satisfies the condition $E_\xi(t) \to 0$ as $t \to \infty$. This yields

$$|I_Q| \leqslant N(n+m) E_\xi(t_1)(t_2 - t_1 + 1) \leqslant$$
$$N(n+m) E_\xi(t_*)(t_2 - t_1 + 1), \quad \tau_\xi + 1 \leqslant t_* < t_1$$

Let us now obtain an estimate for the integral I_R. To do this, we shall divide the segment $[t_1; t_2]$ with the points $\theta_k = t_1 + k\tau$ $(k = 1, 2, \ldots, p-1)$, $\theta_0 = t_1$, $\theta_p = t_2$ into p different subsegments and write $z^{(k)} = z(\theta_k)$. We have the inequality

$$|I_R| \leqslant |F_1| + |F_2|$$

$$F_1 = \sum_{k=0}^{p-1} \int_{\theta_k}^{\theta_{k+1}} \sum_{i=1}^{n+m} \frac{\partial v(t, z(t))}{\partial z_i} R_i(t, z(t))\, dt - F_2$$

$$F_2 = \sum_{k=0}^{p-1} \int_{\theta_k}^{\theta_{k+1}} \sum_{i=1}^{n+m} \frac{\partial v(\theta_k, z^{(k)})}{\partial z_i} R_i(t, z^{(k)})\, dt$$

Since the limit relations (2.2) hold uniformly in $T \in I, z \in B_H \times R^m$, it follows that we can write them in the form

$$\left| \int_t^{t+T} R_i(s, z)\, ds \right| \leqslant \varphi(t) \qquad (i = 1, 2, \ldots n+m),$$

where $\varphi(t)$ is a continuous function monotonically decreasing to zero. This yields the estimates

$$|F_2| \leqslant pN(n+m)\varphi(t_1) < pN(n+m)\varphi(t_*) = pM\gamma$$
$$|F_1| \leqslant pM\tau^2$$

where M is a number depending only on the properties of the function $v(t, z)$ and the constants ε, N, L.

Let Δt be the lower bound of the differences $t_2 - t_1$, such that $\|x(t_1)\| = \xi(\varepsilon), \|x(t_2)\| = \varepsilon$. It is clear that $\Delta t = \Delta t(\varepsilon) > 0$. It what follows, we shall assume that $t_* = t_*(\varepsilon)$ satisfy the inequalities

$$t_* \geqslant \tau_\xi + 1, \quad -c(\xi)/12 + N(n+m) E_\xi(t_*) \leqslant 0$$
$$-c(\xi)\Delta t(\varepsilon)/12 + N(n+m) E_\xi(t_*) < 0 \tag{2.7}$$

Let us choose τ, satisfying the quadratic inequality

$$\tau^2 - 2u\tau + \gamma < 0, \quad u = c(\xi)/(3M).$$

This inequality will hold if $\tau \in]\tau_1, \tau_2[;\ \tau_1 = u - \sqrt{u^2 - \gamma},\ \tau_2 = u + \sqrt{u^2 - \gamma}$ We shall assume the quantity t_* to be so large, that the following inequality holds:

$$\varphi(t_*) < c^2(\xi)/(9MN(n+m)) \tag{2.8}$$

This ensures that the relations $u^2 - \gamma > 0$, $\tau_2 > \tau_1 > 0$ will hold.

We shall show that a natural number p and $\tau \in]\tau_1, \tau_2[$ exist such that the relation

$$p\tau = t_2 - t_1 \tag{2.9}$$

holds. Using condition (2.9), we can write the inequalities $\tau_1 < \tau < \tau_2$ in the form

$$(t_2 - t_1)/\tau_1 > p > (t_2 - t_1)/\tau_2 \tag{2.10}$$

The sufficient condition for the natural number p satisfying the inequality (2.10) to exist is, that

$$\Delta t(\tau_1^{-1} - \tau_2^{-1}) > 1 \tag{2.11}$$

It can be confirmed that inequality (2.11) holds when $\gamma \to 0 + 0$, and this implies that it will also hold when $\gamma < \gamma_1$ where $\gamma_1 = \gamma_1(\varepsilon)$ is sufficiently small. Thus if t_* together with conditions (2.7) and (2.8) satisfy the inequality

$$t_* \geqslant \varphi^{-1}\left(\frac{\gamma_1 M}{N(n+m)}\right)$$

then $\Delta v < -c(\xi)(t_2 - t_1)/6$, and this contradicts relations (2.3). The contradiction shows that for any system of the form (1.5) there are no instants of time $t_1, t_2\ (t_2 > t_1 > t_*(\varepsilon))$, such that $\|x(t_1)\| = \xi$, $\|x(t_2)\| = \varepsilon$.

From the results of /4/ it follows that when conditions (2.1) hold, the estimate $\|X(t, x, y)\| \leqslant$

$L\|x\|$ is valid. Any solution $z(t, t_0, z_0)$ of Eqs.(1.5) depends continuously on the initial conditions, therefore there exists $\delta > 0$ such, that for any $t_0 \in [0; t_*]$, $z_0 \in B_\delta \times B^m$ the condition $\|x(t_*, t_0, z_0)\| < \xi$ holds. Since the quantities ξ and t_* depend only on ε, it follows that δ will also depend only on ε. This proves the uniform x-stability of the solution (1.6) of Eqs.(1.5).

We shall now show that solution (1.6) of system (1.5) is uniformly asymptotically stable with respect to x. Let λ be any fixed number $(0 < \lambda < h)$. We have already proved that there exists $\sigma(\lambda) > 0$ such that any trajectory $z(t) = z(t, t_0, z_0)$ of the system (1.5) satisfying the condition $z_0 \in B_\sigma \times R^m$, will satisfy at any $t > t_0 \geqslant 0$ the condition $z(t) \in B_\lambda \times R^m$.

We shall show that for any $\rho > 0$ $(\rho < \lambda)$ we can show $T(\rho) > 0$ such that the **inequality** $\|x(t)\| < \rho$ holds for any $z_0 \in B_\sigma \times R^m$, $t_0 \in I$, $t \geqslant t_0 + T$. Let $0 < \rho < \lambda$. We have already shown that there exists $\delta(\rho) > 0$ such that condition $z(T_0) \in B_\delta \times R^m$ implies $z(t) \in B_\rho \times R^m$ for any $t \geqslant T_0 \geqslant 0$. Let us estimate the time during which the trajectory may be situated within the region $(B_\lambda \setminus B_\delta) \times R^m$. As before, we can show that

$$\Delta v = v(t, z(t)) - v(T_1, z(T_1)) \leqslant -c(\delta)(t - T_1)/6 \tag{2.12}$$

for $t \geqslant T_1$, where T_1 depends only on $\delta(\rho)$, i.e. $T_1 = T_1(\rho)$. From the inequality (2.12) we obtain

$$t - T_1 \leqslant 6\Delta v / c(\delta) \leqslant$$
$$6[b(\lambda) - a(\delta)]/c(\delta) = T_2(\rho)$$

Putting $T(\rho) = T_1(\rho) + T_2(\rho)$ we find, that for any $z_0 \in B_\sigma \times R^m$, $t_0 \in I$, $t \geqslant t_0 + T$ the inequality $\|x(t, t_0, z_0)\| < \rho$ holds. This means that the solution (1.6) of system (1.5) is uniformly asymptotically stable with respect to x, and the set $B_\sigma \times R^m$ lies within the domain of x-attraction.

Corollary. The conclusions of the theorem remains valid if the function $Q(t, z)$ satisfies the condition A_2 with respect to x, provided that the remaining conditions of the theorem hold. Thus, for example, the condition holds if one of the following relations holds:

$$\lim_{t \to \infty} g(t) = 0, \quad \int^{\infty} g(t)\, dt < \infty$$

We note that in the case when $m = 0$, $R \equiv 0$, the conclusion of the theorem was obtained in /8/, while for $m = 0$, $Q \equiv 0$ the theorem was proved in /9/.

3. We illustrate the application of the theorem by considering the system of ordinary differential Eqs.(1.2) where $n = 2$, $m = 1$, $X_1 = -x_1^3$, $X_2 = -x_2^3$. $Y = x_1 x_2 \sin y$, $R_1^{(1)} = x_2 \cos t^2$, $R_2^{(1)} = x_1 \sin t^2$, $R^{(2)} = x_2 \operatorname{arctg} y \sin t^3$, $Q_1^{(1)} = 0$, $Q_2^{(1)} = x_1(1 + t\,|x_2|)^{-1}$, $Q^{(2)} = 0$. To prove the uniform asymptotic x-stability of the zero solution of system (1.1), we can use the Lyapunov function $v = x_1^2 + x_2^2$. The validity of the condition A_1 with respect to x for the function Q and the feasibility of the limit relations (2.2), can also be proved by elementary methods. Therefore the trivial solution $x_1 = 0$, $x_2 = 0$, $y = 0$ of Eqs.(1.2) is uniformly asymptotically x-stable.

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